J. Fluid Mech. (1994), *vol.* 273, *pp.* 125–139

Slip over rough and coated surfaces

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(Received 21 September 1993 and in revised form 8 February 1994)

We study the effect of surface roughness and coatings on fluid flow over a solid surface. In the limit of small-amplitude roughness and thin lubricating films we are able to derive asymptotically an effective slip boundary condition to replace the no-slip condition over the surface. When the film is absent, the result is a Navier slip condition in which the slip coefficient equals the average amplitude of the roughness. When a layer of a second fluid covers the surface and acts as a lubricating film, the slip coefficient contains a term which is proportional to the viscosity ratio of the two fluids and which depends on the dynamic interaction between the film and the fluid. Limiting cases are identified in which the film dynamics can be decoupled from the outer flow.

1. Introduction

The no-slip boundary condition on the interface between a fluid and a solid is rarely questioned and for a smooth surface is straightforward to apply. For a rough surface, where the scale of the roughness is much smaller than the scale of interest, this is in general a difficult condition to apply. In particular, an accurate numerical solution of the Navier–Stokes equations would require a very fine computational grid in the neighbourhood of the boundary. One means of overcoming this difficulty might be to replace the actual boundary by a smooth surface and hence introduce an effective-slip condition applied on the mean position of the interface. If a Navier slip condition were to emerge in which the slip velocity is proportional to the tangential stress along the surface, the constant of proportionality would be called the slip coefficient.

The use of a slip boundary condition has also been important for the study of the motion of contact lines. It has been shown that if the no-slip condition is used when solving for the motion of a contact line over a solid surface, then a non-integrable rate-of-strain singularity is introduced at the contact line (Dussan V. & Davis 1974; also see Dussan V. 1979 for a review of contact line motion). One means of eliminating the non-integrable singularity is to introduce slip along the boundary (see Dussan V. 1979). Several investigators have studied the effect of roughness in the presence of contact-line motion (Hocking 1976; Huh & Mason 1977; Jansons 1986). There have been several slip models proposed (see Dussan V. 1979): ones with constant slip coefficients and ones where the slip coefficients depend on the thickness of the film. Dussan V. (1976) and Haley & Miksis (1991) found that similar macroscopic behaviour results for all of the local models.

Hocking (1976) studied the effect of the roughness on flow without contact lines. He considered the case of flow over a rough surface where the roughness is modelled as a *periodic* modulation of the surface amplitude. He considered both the case of a single fluid flowing over the surface and the case of one fluid flowing over a rough surface where the grooves are filled with a second fluid and he derived a slip coefficient that includes both effects. Richardson (1971, 1973) has also considered single-phase flow over a rough surface with a periodic modulation of the surface amplitude. By using conformal mapping techniques, he was able to relate the modulation of the surface to the slip coefficient. Jansons (1988) considered the related question of determining the macroscopic slip boundary condition for a viscous fluid flow over a rough surface over which there is perfect slip on the microscale. He found that a very small amount of roughness can induce, a no-slip boundary condition on the macroscopic scale.

Our aim here is to reconsider the roughness problem in the limit of small amplitude but arbitrary-shaped roughness. Here we shall find, if the amplitude of the roughness is small, that the slip coefficient is equal to the average amplitude of the surface roughness. If a second fluid continuously coats the solid surface, then the slip coefficient contains additional terms which depend on the viscosity ratio of the two fluids and the dynamical behaviour of the thin film. In several limiting cases the dynamics of the film decouple from that of the outer fluid, making the concept of a slip boundary condition useful.

2. Formulation

Let the solid/liquid interface be denoted by $y = h(x)$, and assume that the fluid is in the region $y > h(x)$ with $-\infty < x < \infty$, (see figure 1). We assume that the flow far from the surface is prescribed and that the fluid motion is governed by the Navier–Stokes equations. Let Ω_i denote the region containing fluid i of viscosity μ^i and density ρ^i, $i = 1, 2$. Assume that fluid 2 coats the solid, while Ω_1 lies above Ω_2 with the boundary between them being denoted as $y = k(x, t)$. We use superscripts to denote variables in each of the regions and assume that Ω_2 is a thin region, whose thickness is on the same scale as the roughness.

Let U_∞ denote the unit of velocity, which is determined by conditions away from the solid surface. Let L denote the unit of length which represents the macroscopic lengthscale away from the wall and on which we measure the flow field. Let L/U_∞ represent the resultant unit of time and $\mu^1 U_\infty/L$ the unit of pressure. Then in terms of dimensionless variables the equations of motion for the fluids in the region $y > h(x)$ are the conservation of mass

$$\frac{\partial u^i}{\partial x}+\frac{\partial v^i}{\partial y} = 0, \tag{2.1}$$

and the balance of momentum,

$$\hat{\rho}^i Re\left(\frac{\partial u^i}{\partial t}+u^i\frac{\partial u^i}{\partial x}+v^i\frac{\partial u^i}{\partial y}\right)+\frac{\partial p^i}{\partial x} = \hat{\mu}^i\left(\frac{\partial^2 u^i}{\partial x^2}+\frac{\partial^2 u^i}{\partial y^2}\right), \tag{2.2}$$

$$\hat{\rho}^i Re\left(\frac{\partial v^i}{\partial t}+u^i\frac{\partial v^i}{\partial x}+v^i\frac{\partial v^i}{\partial y}\right)+\frac{\partial p^i}{\partial y} = \hat{\mu}^i\left(\frac{\partial^2 v^i}{\partial x^2}+\frac{\partial^2 v^i}{\partial y^2}\right), \tag{2.3}$$

for $i = 1, 2$. Here u represents the x-component of velocity, v represents the y-component of velocity and p represents the pressure. We define the Reynolds number $Re = \rho^1 L U_\infty/\mu_1$, the density ratios $\hat{\rho}^1 = 1$ and $\hat{\rho}^2 = \rho^2/\rho^1$, and the viscosity ratios $\hat{\mu}^1 = 1$ and $\hat{\mu}^2 = \bar{\mu} = \mu^2/\mu^1$.

The solid surface at $y = h(x)$ imposes impenetrability and no-slip:

$$u^2 = 0, \quad v^2 = 0. \tag{2.4}$$

At infinity the velocity field approaches the externally imposed flow. We assume that this external-flow field is consistent with the assumption that the flow fields and their derivatives are bounded in the neighbourhood of the solid surface.

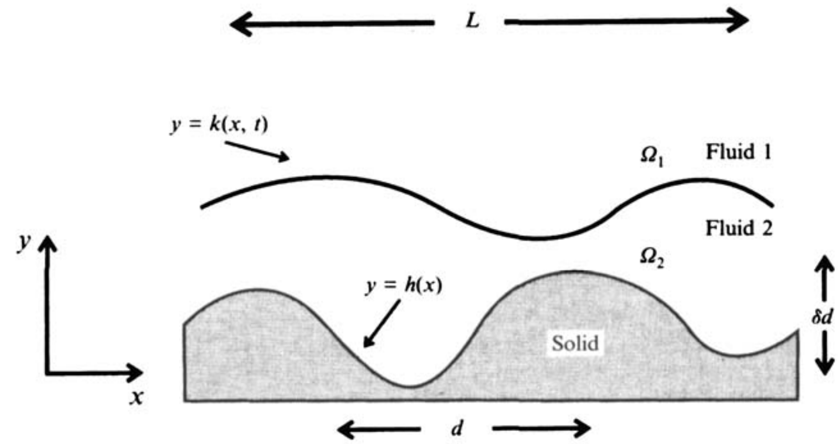

FIGURE 1. Geometry. The rough surface $y = h(x)$, the fluid interface $y = k(x,t)$, and the lengthscales.

Along the free surface $y = k(x,t)$ for $i = 1,2$ we have the kinematic condition

$$\frac{\partial k}{\partial t} + u^i \frac{\partial k}{\partial x} - v^i = 0, \tag{2.5}$$

and the continuity of tangential and normal stresses

$$\sum_{i=1}^{2} (-1)^i \hat{\mu}^i \left\{ k_x \left[\frac{\partial u^i}{\partial x} - \frac{\partial v^i}{\partial y} \right] + \tfrac{1}{2}[k_x^2 - 1] \left[\frac{\partial u^i}{\partial y} + \frac{\partial v^i}{\partial x} \right] \right\} = 0, \tag{2.6}$$

$$p^2 - p^1 + \frac{1}{Ca} \frac{k_{xx}}{(1+k_x^2)^{\frac{3}{2}}} = \sum_{i=1}^{2} (-1)^i \hat{\mu}^i \left[k_x^2 \frac{\partial u^i}{\partial x} + \frac{\partial v^i}{\partial y} - k_x \left(\frac{\partial u^i}{\partial y} + \frac{\partial v^i}{\partial x} \right) \right] (1+k_x^2)^{-1}, \tag{2.7}$$

where the capillary number is $Ca = \mu^1 U_\infty / \sigma$ and σ is the surface tension.

In order to complete the formulation of the problem, we need to specify initial values for all the dependent variables consistent with the no-slip boundary condition (2.4).

The macroscopic lengthscale is L. We assume that the surface roughness is characterized by a much smaller scale d (parallel to the x-axis) and that

$$\epsilon = d/L \ll 1. \tag{2.8}$$

Our aim is to derive an effective boundary condition in the limit of ϵ tending to zero.

Hocking (1976) considered a specific h period in x. Here we let h have a general shape but necessarily small amplitude,

$$y = h(x) = \epsilon \hat{h}(\hat{x}, x), \tag{2.9}$$

where the fast space variable is defined by

$$\hat{x} = x/\epsilon. \tag{2.10}$$

Hence the roughness varies on two scales: the slow scale x and the fast scale $\hat{x}$ over which the amplitude of oscillation is of order ϵ. If the roughness were $\hat{x}$-periodic, then ϵ would represent its wavelength.

The methods of matched asymptotic expansions and multiple scales will be used here to derive the effective-slip boundary condition. The idea is to use the method of matched asymptotics to match the flow field away from the wall with the flow field in the neighbourhood of the wall and then to average out the $\hat{x}$-scale from the results.

Hence, we assume that all variables depend on the fast scale $\hat{x}$ and the two slow scales x and y, e.g. $u = u(\hat{x}, x, y)$. This requires that we replace the x-derivatives in (2.1)–(2.7) by

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \frac{1}{\epsilon}\frac{\partial}{\partial \hat{x}}. \tag{2.11}$$

Hence (2.1) becomes

$$\frac{1}{\epsilon}\frac{\partial u^i}{\partial \hat{x}} + \frac{\partial u^i}{\partial x} + \frac{\partial v^i}{\partial y} = 0. \tag{2.12}$$

Equations (2.2)–(2.7) are transformed in a similar manner. The average $\langle f \rangle$ of a variable $f = f(\hat{x}, x, y, t)$ is defined as

$$\langle f \rangle (x, y, t) = \lim_{W\to\infty} \frac{1}{2W}\int_{x/\epsilon - W}^{x/\epsilon + W} f(\hat{x}, x, y, t)\,\mathrm{d}\hat{x}. \tag{2.13}$$

We assume that all such averages are bounded.

3. Asymptotic solution and the slip boundary condition

We introduce relation (2.11) and apply the method of matched asymptotic expansions to (2.1)–(2.3) along with the boundary conditions (2.4)–(2.7). There are two regions to consider: the outer region, where y is order one, and the inner region, where y is order ϵ.

Outer region. We assume that the interface $y = k(x, t)$ is located in the inner region; hence Ω_2 is a thin film which coats the solid interface. The outer region contains only fluid 1. Look for a solution of (2.1)–(2.3) in the form of a regular perturbation series in ϵ, for example for u^1 we have

$$u^1 = u_0^1(\hat{x}, x, y) + \epsilon u_1^1(\hat{x}, x, y) + \epsilon^2 u_2^1(\hat{x}, x, y) + \dots. \tag{3.1}$$

If we use expansion (3.1) and equate to zero coefficients of like powers of ϵ, we find that the leading-order terms in ϵ satisfy the Navier–Stokes equations (2.1)–(2.3) while the corrections are found by substituting expressions like (3.1) into (2.1)–(2.3) and collecting powers of ϵ. In order for the solution to have a bounded average, no term in the outer expansion can depend on $\hat{x}$ at any order of ϵ. This result implies that the matching condition from the inner region to the outer region cannot depend on the fast scale $\hat{x}$.

Inner region, local to the solid surface. We introduce the inner scaling

$$u^i = \epsilon\hat{u}^i, \quad v^i = \epsilon\hat{v}^i, \quad h = \epsilon\hat{h}^i, \quad y = \epsilon\hat{y}, \quad p^i = \hat{p}^i, \quad k = \epsilon\hat{k}. \tag{3.2}$$

Using expansion (2.11) and scaling (3.2) in (2.1)–(2.6) we find that in the region $\hat{h} < \hat{y} < \infty$ the following equations must hold:

$$\frac{\partial \hat{u}^i}{\partial \hat{x}} + \frac{\partial \hat{v}^i}{\partial \hat{y}} + \epsilon\frac{\partial \hat{u}^i}{\partial x} = 0, \tag{3.3}$$

$$\epsilon^2\hat{\rho}^i Re\left(\frac{\partial \hat{u}^i}{\partial t} + \hat{u}^i\frac{\partial \hat{u}^i}{\partial \hat{x}} + \epsilon\hat{u}^i\frac{\partial \hat{u}^i}{\partial x} + \hat{v}^i\frac{\partial \hat{u}^i}{\partial \hat{y}}\right) + \frac{\partial \hat{p}^i}{\partial \hat{x}} + \epsilon\frac{\partial \hat{p}^i}{\partial x} = \hat{\mu}^i\left(\frac{\partial^2 \hat{u}^i}{\partial \hat{x}^2} + \frac{\partial^2 \hat{v}^i}{\partial \hat{y}^2} + 2\epsilon\frac{\partial^2 \hat{u}^i}{\partial \hat{x}\,\partial x} + \epsilon^2\frac{\partial^2 \hat{u}^i}{\partial x^2}\right), \tag{3.4}$$

$$\epsilon^2\hat{\rho}^i Re\left(\frac{\partial \hat{v}^i}{\partial t} + \hat{u}^i\frac{\partial \hat{v}^i}{\partial \hat{x}} + \epsilon\hat{u}^i\frac{\partial \hat{v}^i}{\partial x} + \hat{v}^i\frac{\partial \hat{v}^i}{\partial \hat{y}}\right) + \frac{\partial \hat{p}}{\partial \hat{y}} = \hat{\mu}^i\left(\frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{y}^2} + 2\epsilon\frac{\partial^2 \hat{v}}{\partial \hat{x}\,\partial x} + \epsilon^2\frac{\partial^2 \hat{v}}{\partial x^2}\right) \tag{3.5}$$

for $i = 1, 2$. We note that, if there were no $\hat{x}$-dependence in (3.3)–(3.5) and $\epsilon \to 0$, then these would be the lubrication equations governing the liquid film.

In addition we have from (2.4) that the boundary conditions

$$\hat{u}^2 = \hat{v}^2 = 0 \tag{3.6}$$

must hold on $\hat{y} = \hat{h}(\hat{x}, x)$. Finally, the boundary conditions (2.5)–(2.7) along the interface $\hat{y} = \hat{k}(x, \hat{x}, t)$ are

$$\frac{\partial \hat{k}}{\partial t} + \hat{u}^i(\hat{k}_{\hat{x}} + \epsilon \hat{k}_x) - \hat{v}^i = 0, \tag{3.7}$$

$$\sum_{i=1}^{2} (-1)^i \hat{\mu}^i \left\{ (\hat{k}_{\hat{x}} + \epsilon \hat{k}_x) \left[\frac{\partial \hat{u}^i}{\partial \hat{x}} + \epsilon \frac{\partial \hat{u}^i}{\partial x} - \frac{\partial \hat{v}^i}{\partial \hat{y}} \right] + \tfrac{1}{2}[(\hat{k}_{\hat{x}} + \epsilon \hat{k}_x)^2 - 1] \left[\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} + \epsilon \frac{\partial \hat{v}}{\partial x} \right] \right\} = 0, \tag{3.8}$$

$$\hat{p}^2 - \hat{p}^1 + \frac{1}{\widehat{Ca}} \frac{\hat{k}_{xx}}{[1 + (\hat{k}_{\hat{x}} + \epsilon \hat{k}_x)^2]^{\frac{3}{2}}} = \left\{ \sum_{i=1}^{2} (-1)^i \hat{\mu}^i \left[(\hat{k}_{\hat{x}} + \epsilon \hat{k}_x)^2 \left(\frac{\partial \hat{u}^i}{\partial \hat{x}} + \epsilon \frac{\partial \hat{u}^i}{\partial x} \right) + \frac{\partial \hat{v}^i}{\partial \hat{y}} - (\hat{k}_{\hat{x}} + \epsilon \hat{k}_x) \left(\frac{\partial \hat{u}^i}{\partial \hat{y}} + \frac{\partial \hat{v}^i}{\partial \hat{x}} + \epsilon \frac{\partial \hat{v}^i}{\partial x} \right) \right] \right\} [1 + (\hat{k}_{\hat{x}} + \epsilon \hat{k}_x)^2]^{-1}, \tag{3.9}$$

where we have defined the scaled capillary number, $\widehat{Ca} = \epsilon Ca$. Solutions of system (3.3)–(3.9) are found as a regular perturbation series in ϵ, e.g.

$$\hat{u} = \hat{u}^0(\hat{x}, x, \hat{y}) + \epsilon \hat{u}^1(\hat{x}, x, \hat{y}) + \epsilon^2 \hat{u}^2(\hat{x}, x, \hat{y}) + \dots. \tag{3.10}$$

We substitute expansion (3.10) into (3.3)–(3.5) and equate to zero coefficients of like powers of ϵ. At leading order we find that $\hat{u}_0$, $\hat{v}_0$ and $\hat{p}_0$ satisfy the Stokes equations, i.e.

$$\frac{\partial \hat{u}_0^i}{\partial \hat{x}} + \frac{\partial \hat{v}_0^i}{\partial \hat{y}} = 0, \tag{3.11}$$

$$\frac{\partial \hat{p}_0^i}{\partial \hat{x}} = \hat{\mu}^i \left(\frac{\partial^2 \hat{u}_0^i}{\partial \hat{x}^2} + \frac{\partial^2 \hat{u}_0^i}{\partial \hat{y}^2} \right), \tag{3.12}$$

$$\frac{\partial \hat{p}_0^i}{\partial \hat{y}} = \frac{\partial^2 \hat{v}_0^i}{\partial \hat{x}^2} + \frac{\partial^2 \hat{v}_0^i}{\partial \hat{y}^2}. \tag{3.13}$$

If (3.11)–(3.13) were independent of $\hat{x}$, for example if no roughness were present, then it would represent the lubrication equations. The equations for the higher-order terms as well as the leading-order boundary conditions can be found in a similar manner.

This problem is as difficult to solve as the original one except for the fact that the roughness is of unit order in the inner region.

Suppose we average (3.11)–(3.13). Assume that we can interchange the $\hat{y}$-derivative with the average, that the dependent variables are bounded for all $\hat{x}$, and that $\hat{y} > \hat{k}$ for all x and $\hat{x}$, i.e. the lines $\hat{y}$ is constant are completely in Ω_1. We then find that both $\langle \hat{v}_0^1 \rangle$ and $\langle \hat{p}_0^1 \rangle$ are independent of $\hat{y}$ while $\langle \hat{u}_0^1 \rangle$ is a linear function of $\hat{y}$,

$$\langle \hat{u}_0^1 \rangle (x, \hat{y}, t) = a(x, t) + b(x, t)\, \hat{y}. \tag{3.14}$$

Here a and b are two functions of which must be determined by the solution of the inner problem and matching. To leading order the matching condition from the outer region to the inner region is given by

$$u_0^1(x,0,t) = v_0^1(x,0,t) = 0. \tag{3.15}$$

Hence the leading-order outer solution can be completely determined. This requires that the Navier–Stokes equations along a planar no-slip boundary be solved for u_0^1, v_0^1 and p_0^1. Initial conditions plus boundary conditions at infinity must be specified. We assume that the conditions on the outer flow are such that spatial derivatives of the dependent variables are finite along $y = 0$, for example, u_0^1 has a regular Taylor series expansion about $y = 0$. Suppose that the conditions are such that the outer flow is a steady linear shearing flow, then $u_0^1 = y$, $v_0^1 = 0$ and $p^0 =$ constant are the solutions which match to the shearing flow $u^1 = y$ and $v^1 = 0$ far from the boundary.

Continuing with the matching, we find that in general the leading-order inner solution must match to a linear shearing flow at infinity. Hence we find that $b(x,t) = (\partial u_0^1/\partial y)(x,0,t)$ and we rewrite (3.14) as

$$\langle \hat{u}_0^1 \rangle (x,\hat{y},t) = [\hat{y} - \hat{C}(x,t)] \frac{\partial u_0^1}{\partial y}(x,0,t). \tag{3.16}$$

Here we have introduced the function $\hat{C}(x,t)$ defined by $a(x,t) = -\hat{C}(x,y)(\partial u_0^1/\partial y)(x,0,t)$. It depends on the amplitude of the roughness $h(\hat{x},x,t)$ and the shear at infinity. The next-order matching condition for the outer solution could be found by using (3.16) since the outer solution cannot depend on $\hat{x}$. It is

$$u_1^1(x,0,t) = -\hat{C}(x,t)\frac{\partial u_0^1}{\partial y}(x,0,t). \tag{3.17}$$

If we multiply (3.17) by ϵ and add (3.15) at $y = 0$, we find that to order ϵ the outer flow satisfies the effective boundary condition along $y = 0$,

$$u(x,0,t) + C(x,t)\frac{\partial u}{\partial y}(x,0,t) = 0, \tag{3.18}$$

where $C(x,t) = \epsilon\hat{C}(x,t)$. This is a Navier slip condition with a slip coefficient $C(x,t)$. Hence to order ϵ, in the outer region we must solve the Navier–Stokes equations (2.1)–(2.3) with the Navier slip condition (3.18). In addition, as we shall show, the flux boundary condition is $v(x,0,t) = 0$ to order ϵ. At this point we only know that $v = 0$ to leading order; see (3.15). Our aim in the following is to determine $C(x,t)$ in the limit of small-amplitude roughness.

4. Inner solution

In order to solve (3.11)–(3.13) we assume that the amplitude of the roughness is small in the inner region. This requires the introduction of a second small parameter δ and we set $\hat{h} = \delta\bar{h}(\hat{x},x)$, where $\bar{h}$ is assumed to be order one. Our final result will be valid to order $\epsilon\delta$, hence we need that

$$\epsilon \ll \delta \ll 1. \tag{4.1}$$

For a smaller δ we would need to consider additional terms in the ϵ-expansion.

Now apply the method of matched asymptotic expansions once again in order to solve the inner problem (3.11)–(3.13). We need to again consider two regions: the first is where $\hat{y}$ is of unit order – call this region I; and the second is where $\hat{y}$ is of order δ – call this region II. In region I we seek a solution in the form of a regular perturbation series in δ,

$$\left.\begin{aligned} \hat{u}_0^1 &= \hat{U}_0(\hat{x},x,\hat{y},t)+\delta\hat{U}_1(\hat{x},x,\hat{y},t)+\delta^2\hat{U}_2(\hat{x},x,\hat{y},t)+\ldots,\\ \hat{v}_0^1 &= \hat{V}_0(\hat{x},x,\hat{y},t)+\delta\hat{V}_1(\hat{x},x,\hat{y},t)+\delta^2\hat{V}_2(\hat{x},x,\hat{y},t)+\ldots,\\ \hat{p}_0^1 &= \hat{P}_0(\hat{x},x,\hat{y},t)+\delta\hat{P}_1(\hat{x},x,\hat{y},t)+\delta^2\hat{P}_2(\hat{x},x,\hat{y},t)+\ldots. \end{aligned}\right\} \tag{4.2}$$

We substitute expansions (4.2) into (3.11)–(3.13) and find that at each order the Stokes equations result.

We wish to consider two limits of the solution in region II. The first applies where there is no coating fluid. The second applies when the coating fluid exists.

4.1. *Single-phase flow over a rough surface*

For this case there is no coating fluid and so there is no region Ω_2 and no interface. The velocity components $\hat{u}_0^1$ and $\hat{v}_0^1$ satisfy the no-slip condition along the solid surface $\hat{y} = \hat{h}$.

We rescale the variables in region II as

$$\hat{u}_0^1 = \delta\bar{u}^1, \quad \hat{v}_0^1 = \delta^2\bar{v}^1, \quad \hat{p}_0^1 = \bar{p}^1, \quad \hat{y} = \delta\bar{y}, \quad \hat{h} = \delta\bar{h}, \tag{4.3}$$

and again we seek a solution as a power series in δ, e.g.

$$\bar{u}^1 = \bar{u}_0^1(\hat{x},x,\bar{y},t)+\delta\bar{u}_1^1(\hat{x},x,\bar{y},t)+\delta^2\bar{u}_2^1(\hat{x},x,\bar{y},t)+\ldots. \tag{4.4}$$

By substituting (4.3)–(4.4) into (3.11)–(3.13) we find that the leading-order equations are

$$\frac{\partial\bar{u}_0^1}{\partial\hat{x}}+\frac{\partial\bar{v}_0^1}{\partial\bar{y}} = 0, \tag{4.5}$$

$$\frac{\partial^2\bar{u}_0^1}{\partial\bar{y}^2} = 0, \tag{4.6}$$

$$\frac{\partial\bar{p}_0^1}{\partial\bar{y}} = 0. \tag{4.7}$$

These equations are to be solved with the no-slip boundary conditions, which at leading order in δ, are

$$\bar{u}_0^1 = 0, \quad \bar{v}_0^1 = 0 \tag{4.8}$$

along $\bar{y} = \bar{h}(\hat{x},x)$. The solution of (4.5)–(4.8) is

$$\bar{u}_0^1 = (\bar{y}-\bar{h})\,K. \tag{4.9}$$

$$\bar{v}_0^1 = \tfrac{1}{2}[\bar{h}^2-\bar{y}^2]\frac{\partial K}{\partial\hat{x}}-(\bar{h}-\bar{y})\frac{\partial}{\partial\hat{x}}(\bar{h}K), \tag{4.10}$$

$$\bar{p}_0^1 = \bar{p}_0^1(\hat{x},x,t). \tag{4.11}$$

Here the unknown function of integration $K = K(\hat{x},x,t)$ will be determined by matching. Higher-order terms are determined in a similar manner.

We wish to match the solutions in region I and II. Hence, we rewrite the solution in region I in terms of the variable $\bar{y} = \delta\hat{y}$ and expand the dependent variables about $\hat{y} = 0$. We find that the matching conditions along $\hat{y} = 0$ are

$$\hat{U}_0(\hat{x}, x, 0, t) = 0, \quad \hat{U}_1(\hat{x}, x, 0, t) = -\bar{h}K, \quad \frac{\partial \hat{U}_0}{\partial \hat{y}}(\hat{x}, x, 0, t) = K, \tag{4.12}$$

and also along $\hat{y} = 0$ we find for $\hat{v}_0^1$ that

$$\hat{V}_0 = \hat{V}_1 = \frac{\partial \hat{V}_0}{\partial \hat{y}} = 0. \tag{4.13}$$

This immediately gives that the leading-order solution in region I is a linear shearing flow

$$\hat{U}_0 = \hat{y}\frac{\partial u_0^1}{\partial y}(x, 0, t), \quad \hat{V}_0 = 0. \tag{4.14}$$

The leading-order pressure $\hat{P}_0$ is independent of $\hat{x}$ and $\hat{y}$ and equals the outer pressure, $\hat{P}_0 = p_0^1$. From (4.12) we find that K is equal to $\partial u_0^1/\partial y$ at $y = 0$. We also note from relation (4.10) that the quadratic term in the inner velocity v_0^1 drops out.

Now the next-order matching condition for the velocity $\hat{U}_1$ from (4.12) is that $\hat{U}_1(\hat{x}, x, 0, t) = -\bar{h}(\hat{x}, x)(\partial u_0^1/\partial y)(x, 0, t)$. Since the leading-order term $\hat{U}_0$ matches to the linear-shear velocity as $\hat{y}$ tends to infinity, we now require that $\hat{U}_1$ be bounded as $\hat{y}$ tends to infinity. Hence, the solution of the order-δ problem away from the wall must satisfy these boundary conditions plus the Stokes equations. This is a difficult problem but as noted above we only need to have the average behaviour of $\hat{U}_1$. Therefore, if we average the system as discussed earlier, we find that $\langle \hat{U}_1 \rangle$ is linear in $\hat{y}$. In order to satisfy the above boundary conditions we find that

$$\langle \hat{U}_1 \rangle = -\langle \bar{h} \rangle \frac{\partial u_0^1}{\partial y}(x, 0, t), \quad \langle \hat{V}_1 \rangle = 0. \tag{4.15}$$

This result, along with (3.16), (4.2) and (4.14), implies that the slip coefficient is

$$C(x) = \langle h \rangle, \tag{4.16}$$

and the effective boundary condition along $y = 0$, to order $\epsilon\delta$, for flow over a rough surface is

$$u(x, 0, t) + \langle h \rangle \frac{\partial u}{\partial y}(x, 0, t) = 0. \tag{4.17}$$

Further, from (4.15) we validate the remark made at the end of §3 that the flux boundary condition is $v = 0$ at $y = 0$ to order $\epsilon\delta$. Note that order-$\epsilon\delta^2$ terms have been used in the matching, but results accurate to order $\epsilon\delta$ are used in the slip condition (4.17) and the boundary condition on v. Hence here, as well as in §4.2, condition (4.1) is sufficient to give order-$\epsilon\delta$ accuracy of the effective boundary conditions.

The position of the origin of the coordinate system relative to the modulation of the solid surface will affect the predicted slip coefficient. For example, if we simply put the origin at the location where $\langle h \rangle = 0$, then to order $\epsilon\delta$ the slip coefficient is zero. Hocking (1976) assumed that the surface $y = h$ is located below $y = 0$ with the effective plane coinciding with the maximum of h. Under this assumption, (4.17) reproduces Hocking's result for a small-amplitude periodic modulation.

The dependence of the slip coefficient on the location of the origin of the coordinate system reflects the fact that the solution of a flow problem *should* depend on the size of the domain, and if the roughness has a small amplitude, then the slip coefficient can be varied by the position of the coordinate plane within this order-$\epsilon\delta$ region. In any real problem, the location of the origin and the size of the domain is fixed first, and then the amplitude of the roughness is measured relative to the origin chosen.

In the next section, following Hocking (1976), we shall consider that the solid surface is coated by a second fluid. Here we shall find that slip coefficient depends on the material properties of the second fluid and this contribution can dominate the dependence on the average roughness.

4.2. *Two-phase flow over a rough surface*

There is now a coating fluid along the solid surface. In order to solve the inner problem we again assume that the amplitude of the surface roughness in the inner region is of order δ and that the thickness of region Ω_2 is also of order δ. Hence, in region II we need to rescale the variables as

$$\hat{u}_0^i = \delta\bar{u}^i, \quad \hat{v}_0^i = \delta^2\bar{v}^i, \quad \hat{p}_0^i = \delta^{-1}\bar{p}^i, \quad \hat{y} = \delta\bar{y}, \quad \hat{h} = \delta\bar{h}, \quad \hat{k} = \delta\bar{k} \tag{4.18}$$

for $i = 1, 2$, and again seek a solution as a power series in δ; see (4.4). Note that we have increased the pressure scale in order to allow for the possibility of large surface tension. Note also that a new timescale has been implicitly introduced into this problem as a result of the new scales (4.18), defined by $T = \delta t$. Hence our problem now has two timescales: t, the (fast) time over which the macroscopic flow is being driven far away from the solid surface; and T, the slow timescale over which phenomena local to the wall are adjusting to the forcing of the flow field far from the solid surface. Hence, analogous to the separation of the two space scales, x and $\hat{x}$, we now look for all dependent variables as a function of both t and T and replace the t-derivatives in the equations of motion by

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \delta\frac{\partial}{\partial T}. \tag{4.19}$$

This substitution (4.19) is made into the evolution equations and boundary conditions (2.1)–(2.7). We also assume that the macroscopic forcing in the problem contains both of these timescales. Hence the leading-order outer problem in ϵ will depend on both t and T, for example $u_0^1 = u_0^1(x, y, T, t)$.

We substitute (4.18), (4.19) and (4.4) into (3.11)–(3.13) and find that the leading-order equations in region II are

$$\frac{\partial\bar{u}_0^i}{\partial\hat{x}} + \frac{\partial\bar{v}_0^i}{\partial\bar{y}} = 0, \tag{4.20}$$

$$\frac{\partial\bar{p}_0^i}{\partial\hat{x}} = \hat{\mu}^i\frac{\partial^2\bar{u}_0^i}{\partial\bar{y}^2}, \tag{4.21}$$

$$\frac{\partial\bar{p}_0^i}{\partial\bar{y}} = 0 \tag{4.22}$$

for $i = 1, 2$. These equations are to be solved with the no-slip boundary conditions along $\bar{y} = \bar{h}(\hat{x}, x)$,

$$\bar{u}_0^2 = 0, \quad \bar{v}_0^2 = 0. \tag{4.23}$$

Along the interface $\bar{y} = \bar{k}_0(\hat{x}, x, t)$ the leading-order boundary conditions from (3.7)–(3.9) are

$$\frac{\partial \bar{k}_0}{\partial t} = 0, \tag{4.24}$$

$$\frac{\partial \bar{u}_0^1}{\partial \bar{y}} = \bar{\mu}\frac{\partial \bar{u}_0^2}{\partial \bar{y}}, \tag{4.25}$$

$$\bar{p}_0^2 - \bar{p}_0^1 + \frac{1}{\overline{Ca}}\bar{k}_{0\hat{x}\hat{x}} = 0 \tag{4.26}$$

for $i = 1, 2$. Here we have defined the scaled capillary number $\overline{Ca} = \epsilon Ca/\delta^2$ and assume that it is order one. In addition we have continuity of velocity across $\bar{y} = \bar{k}_0$,

$$\bar{u}_0^1 = \bar{u}_0^2, \quad \bar{v}_0^1 = \bar{v}_0^2. \tag{4.27}$$

Note that the scaling has given us a system of equations similar to the lubrication limit for thin film flow over a rough surface. The difference occurs only in the kinematic condition along the interface which will now have an additional time dependence.

Note that the next-order kinematic condition along $\bar{y} = \bar{k}_0(\hat{x}, x, T, t)$ is

$$\frac{\partial \bar{k}_1}{\partial t} + \frac{\partial \bar{k}_0}{\partial T} + \bar{u}_0^i \bar{k}_{0\hat{x}} - \bar{v}_0^i = 0. \tag{4.28}$$

As for the case of the two space scales, we require that the average over the fast timescale t of all variables is bounded in time. The average $\tilde{f}$ of $f(\hat{x}, x, y, T, t)$ is defined as

$$\tilde{f}(\hat{x}, x, y, T) = \lim_{W \to \infty} \frac{1}{W}\int_0^W f(\hat{x}, x, y, T, t)\,\mathrm{d}t. \tag{4.29}$$

The matching condition on the pressure requires that $\bar{p}_0^1 = 0$. Hence the leading-order effect on the pressure in Ω_2 is due only to the effect of capillarity. From (4.20)–(4.21) we then find that the leading-order velocities are given by

$$\bar{u}_0^1 = A\bar{y} + B, \tag{4.30}$$

$$\bar{v}_0^1 = -\tfrac{1}{2}A_{\hat{x}}\bar{y}^2 - B_{\hat{x}}\bar{y} - D, \tag{4.31}$$

where the functions $A(\hat{x}, x, T, t)$, $B(\hat{x}, x, T, t)$ and $D(\hat{x}, x, T, t)$ are yet to be determined. In §3 we noted that the $\hat{u}_0^1$-velocity must match to a linear shearing flow at infinity. Hence $\hat{U}_0$ matches to the same shearing velocity at infinity, $\hat{y}(\partial u_0^1/\partial y)(x, 0, T, t)$. Therefore, for the velocity from region II to match to the velocity in region I and satisfy the above matching conditions we need that $A = (\partial u_0^1/\partial y)(x, 0, T, t)$; hence A is independent of $\hat{x}$ and the presence of the boundary will affect only the coefficient B. Following a similar calculation as done in §3, we are able to relate the average of B to the slip coefficient.

We integrate (4.20)–(4.22) and use the boundary conditions (4.23) to find that in Ω_2

$$\bar{u}_0^2 = \frac{1}{2\bar{\mu}}q_{\hat{x}}(\bar{y}^2 - \bar{h}^2) + K(\bar{y} - \bar{h}), \tag{4.32}$$

$$\bar{v}_0^2 = -\frac{1}{6\bar{\mu}}\frac{\partial}{\partial \hat{x}}[q_{\hat{x}}(\bar{y}^3 - 3\bar{h}^2\bar{y} + 2\bar{h}^3)] - \frac{1}{2}\frac{\partial}{\partial \hat{x}}[K(\bar{y} - \bar{h})^2], \tag{4.33}$$

$$\bar{p}_0^2 = q(\hat{x}, x, t), \tag{4.34}$$

where the unknown functions $q(\hat{x}, x, t)$ and $K(\hat{x}, x, t)$ are determined by the boundary and matching conditions. Using (4.30) and (4.32) in (4.25), we find that $K = (A - q_{\hat{x}}\bar{k}_0)/\bar{\mu}$. The unknown function q can be related to $\bar{k}_0$ using (4.26) to find that $q = -\bar{k}_{0\hat{x}\hat{x}}/\overline{Ca}$.

The kinematic condition determines the interface position $\bar{y} = \bar{k}_0$. First, from (4.24) we find that $\bar{k}_0$ is independent of the fast timescale t. Hence using the above results along with (4.32) and (4.33), we find from (4.28) that

$$\frac{\partial \bar{k}_1}{\partial t} = -\left\{\frac{\partial \bar{k}_0}{\partial T} + \frac{1}{3\bar{\mu}\overline{Ca}}\frac{\partial}{\partial \hat{x}}[\bar{k}_{0\hat{x}\hat{x}\hat{x}}(\bar{k}_0 - \bar{h})^3] + \frac{1}{2\bar{\mu}}\frac{\partial}{\partial \hat{x}}[A(\bar{k}_0 - \bar{h})^2]\right\}. \tag{4.35}$$

where, as noted above, $A = (\partial u_0^1/\partial y)(x, 0, T, t)$. In order for a solution of (4.35) to exist, we need to require that the time average of the right-hand side be bounded. We find that a solution exists only if $\bar{k}_0$ satisfies the equation

$$\bar{\mu}\frac{\partial \bar{k}_0}{\partial T} = -\frac{1}{3\overline{Ca}}\frac{\partial}{\partial \hat{x}}[\bar{k}_{0\hat{x}\hat{x}\hat{x}}(\bar{k}_0 - \bar{h})^3] - \frac{1}{2}\frac{\partial}{\partial \hat{x}}[\tilde{A}(\bar{k}_0 - \bar{h})^2]. \tag{4.36}$$

Hence, the leading-order dynamics of the interface is governed by the evolution equation (4.36) and changes to $\bar{k}_0$ occur on the slow timescale T. Note that the dynamics depends on the coupling to the macroscopic flow field through the term $\tilde{A}$ which represents the shear stress that the outer fluid exerts on the interface. Only if there were a slow external forcing of the macroscopic flow would this term depend on T.

Suppose that we average (4.36) over the $\hat{x}$-scale. Interchanging the T-derivative with the average implies that $\langle \bar{k}_0 \rangle$ depends only on x, i.e. the leading-order average thickness of the film is constant in time (conservation of mass on the microscopic scale). Hence although there can be a local rearrangement of the fluid in the film, as given by (4.36), the average film thickness does not change in time at leading order in this small-scale roughness limit. It can be expected that over a very long timescale this would not be the case and hence our asymptotic expansion would break down; corrections to the theory would then be needed.

The function B can now be determined by using (4.30) and (4.32) in (4.27),

$$B(\hat{x}, x, T, t) = -A\bar{h} + A(\bar{k}_0 - \bar{h})\left(\frac{1}{\bar{\mu}} - 1\right) + \frac{1}{2\bar{\mu}\overline{Ca}}(\bar{k}_0 - \bar{h})^2\frac{\partial^3 \bar{k}_0}{\partial \hat{x}^3}. \tag{4.37}$$

The problem in region I can be solved by matching $\hat{U}_0$ to $\bar{u}_0^1$, as given by (4.30), and $\hat{V}_0$ to zero at $\hat{y} = 0$. The problem reduces to the solution of the Stokes equations with these conditions at $\hat{y} = 0$ and a linear shearing flow at infinity. This is a difficult problem but as noted in §4.1 we need only solve for the average values of the unknowns. This allows us to determine the scaled slip coefficient $\bar{C}(x, T, t) = C/\delta\epsilon$ as

$$\bar{C}(x, T, t) = \langle \bar{h} \rangle - \left(\frac{1}{\bar{\mu}} - 1\right)\bar{D}(x) - \frac{1}{2\bar{\mu}A\overline{Ca}}\left\langle (\bar{k}_0 - \bar{h})^2\frac{\partial^3 \bar{k}_0}{\partial \hat{x}^3}\right\rangle, \tag{4.38}$$

where we have defined the average film thickness $\bar{D}(x) = \langle \bar{k}_0 - \bar{h} \rangle$. Hence, the average depth does not depend on the time. The dependence on the fast time t enters (4.38) only in the third term with the factor A. The unscaled slip coefficient $C(x, t)$ (in outer units) can be written as follows:

$$C(x, T, t) = \langle h \rangle - \left(\frac{\hat{\mu}^1}{\hat{\mu}^2} - 1\right)D(x) - \frac{\hat{\mu}^1}{2\hat{\mu}^2 A Ca}\left\langle (k - h)^2\frac{\partial^3 k}{\partial \hat{x}^3}\right\rangle, \tag{4.39}$$

where $D(x) = \langle k-h \rangle$ and k represents the leading-order contribution to the interface height.

Note that A can be replaced by $\partial u/\partial y$ in (4.39) and C will retain the same order of accuracy. The substitution shows explicitly how the outer flow can couple to the microscopic flow field but it does not, in general, aid in the calculation of the slip coefficient since the microscopic flow is required in order to calculate the averages in (4.39). In the next section we shall discuss certain limiting cases where the calculation of the slip coefficient decouples from the macroscopic flow field.

Note that in the derivation of the slip coefficient, A has been assumed to be non-zero. If A were zero, the derivation would break down locally since the outer flow would be separating, $\partial u/\partial y = 0$ at $y = 0$.

5. The slip coefficient

In order to better understand the behaviour of the slip coefficient, we consider in this section some special limits of (4.38) or (4.39). Note that if there were no coating film, i.e. $k = h$ and $\mu^1 = \mu^2$, then (4.39) would reproduce (4.16), the slip coefficient for the single-phase problem.

Recall that $\overline{Ca}$ equals Ca times the ratio ϵ/δ^2. This later ratio of dimensionless parameters can be either large or small depending on the relation of δ to ϵ. Hence, the geometry, as well as the physical parameters (e.g. surface tension) of the fluids, determine the magnitude of the scaled capillary number $\overline{Ca}$.

Large capillary number, $\overline{Ca}$

Suppose that the capillary number is large, $\overline{Ca} \gg 1$. Then we can neglect the third term in form (4.38) and the slip coefficient is given by

$$C = \langle h \rangle - \left(\frac{\hat{\mu}^1}{\hat{\mu}^2} - 1\right) d(x). \tag{5.1}$$

Hence, C is independent of time. If surface roughness were absent or if $\hat{\mu}^1/\hat{\mu}^2 \gg 1$, the slip coefficient would be proportional to the mean thickness of the coating film. Note that the slip coefficient is straightforward to calculate in this large-capillary-number limit and is given by the initial data. Although in the general unsteady case, the dynamics of the interface is coupled to the macroscopic flow problem by the slip coefficient, when $\overline{Ca} \gg 1$ only the mean thickness of the thin film enters.

Small capillary number, $\overline{Ca}$

Suppose that the capillary number is small, $\overline{Ca} \ll 1$. This could represent the large-surface-tension limit or just a small ϵ/δ^2 ratio. If we were to look for a solution of the leading-order equations as a regular perturbation expansion in $\overline{Ca}$, then from (4.36) the leading-order term of the expansion must have either $\bar{k}_{0\hat{x}\hat{x}\hat{x}} = 0$ or $\bar{k}_0 = \bar{h}$. Suppose we assume that there is sufficient fluid in the film such that the latter option is not possible (i.e. the volume in the thin film per unit length is of unit order) so that we must have $\bar{k}_0 = \bar{k}_0(x)$, a function independent of $\hat{x}$ and T at leading order in $\overline{Ca}$. In order to determine C one must continue to next order in $\overline{Ca}$ and obtain

$$C(x,t) = \langle h \rangle - \left(\frac{\mu^1}{\mu^2} - 1\right) D(x,T) + \frac{\mu^1}{3\mu^2}\frac{\tilde{A}}{A}\left[\langle k-h \rangle - \left\langle \frac{1}{k-h} \right\rangle^2 \left\langle \frac{1}{(k-h)^3} \right\rangle^{-1}\right], \tag{5.2}$$

where $k = k(x)$ to leading order in $\overline{Ca}$ (here we replace $\bar{k}_0$ by k). Note that if h varies only on the long scale x, then (5.2) reduces to (5.1). Also note that if the external flow does not depend on t, then A does not appear in (5.2) and the slip coefficient is given explicitly in terms of the surface roughness and the initial (constant on the $\hat{x}$-scale) film thickness.

Small film viscosity, $\bar{\mu} \ll 1$

Another interesting limit is where $\bar{\mu} = \mu^2/\mu^1 \ll 1$, i.e. the coating fluid is much less viscous that the fluid far from the solid surface as would be the case if a layer of gas were trapped below a liquid. Here we find that to leading order in $\bar{\mu}$, the slip coefficient is independent of the mean surface roughness amplitude. We see that the leading-order behaviour of the interface is now determined by setting the left-hand side of (4.36) to zero. Hence $\bar{k}_0$ is determined by solving an ordinary differential equation, the quasi-steady form of (4.36). After one integration we find that

$$\tilde{A} = -\frac{2}{3\overline{Ca}}(\bar{k}_0 - \bar{h})\,\bar{k}_{0\hat{x}\hat{x}\hat{x}} + L(x, T)\,(\bar{k}_0 - \bar{h})^{-2}, \tag{5.3}$$

where the function $L(x, T)$ is chosen to ensure mass conservation and the boundness of the $\hat{x}$-average of $\bar{k}$. Hence, at leading-order, the interface is constant in time, and is given by the initial conditions unless A and the leading-order macroscopic flow field varies on the T timescale. If $\overline{Ca}$ is also large, then as noted above, the slip coefficient would depend only on the product of the viscosity ratio with the mean film thickness.

Large film viscosity, $\bar{\mu} \gg 1$

Suppose we consider the limit where $\bar{\mu} = \mu^2/\mu^1 \gg 1$, i.e. the coating fluid is much more viscous than the fluid in Ω_1. Again from (4.36) we find that at leading-order, $\bar{k}_0$ is independent of T. From (5.1) we see that in the large-viscosity-ratio limit, $\bar{\mu} \gg 1$, the leading-order slip coefficient is $C = \langle \bar{k}_0 \rangle$. Hence the slip coefficient is known explicitly in terms of the initial data. The coating fluid acts as a solid in this limit and the leading-order value of the slip coefficient just corrects from the error resulting from applying the no-slip boundary condition at the bottom of the coating film. This calculation is valid over the timescale T, but for larger times one would need to introduce an additional timescale $T/\bar{\mu}$ into the problem in order to avoid secular terms in time in the expansion of $\bar{k}_0$.

Smooth surface, $\langle h \rangle = 0$

Finally note that (4.38) also holds in the case of a smooth surface, $\langle h \rangle = 0$, with a coating fluid. In this limit only the last two terms of (4.38) are non-zero. Hence, as noted above, in the limit where $\mu^1/\mu^2 \gg 1$, we find that the slip coefficient depends only on the product of the viscosity ratio and the mean thickness of the coating film.

6. Conclusions

Our aim in this paper is to systematically derive an effective boundary condition for flow over a rough or coated surface. For a single-phase flow the result is a Navier slip condition and the slip coefficient (4.16) is equal to the average amplitude of the roughness, $\langle h \rangle$. As noted in §4, the actual numerical value of this will depend on the origin of the coordinate system. In the two-phase flow case the slip coefficient is given by (4.38). Hence, the solution of the Navier–Stokes equations for flow over a coated, rough surface (figure 1), can be replaced by the solution of a single-phase flow over a smooth surface (figure 2*a*) when slip over the surface is governed by a Navier slip

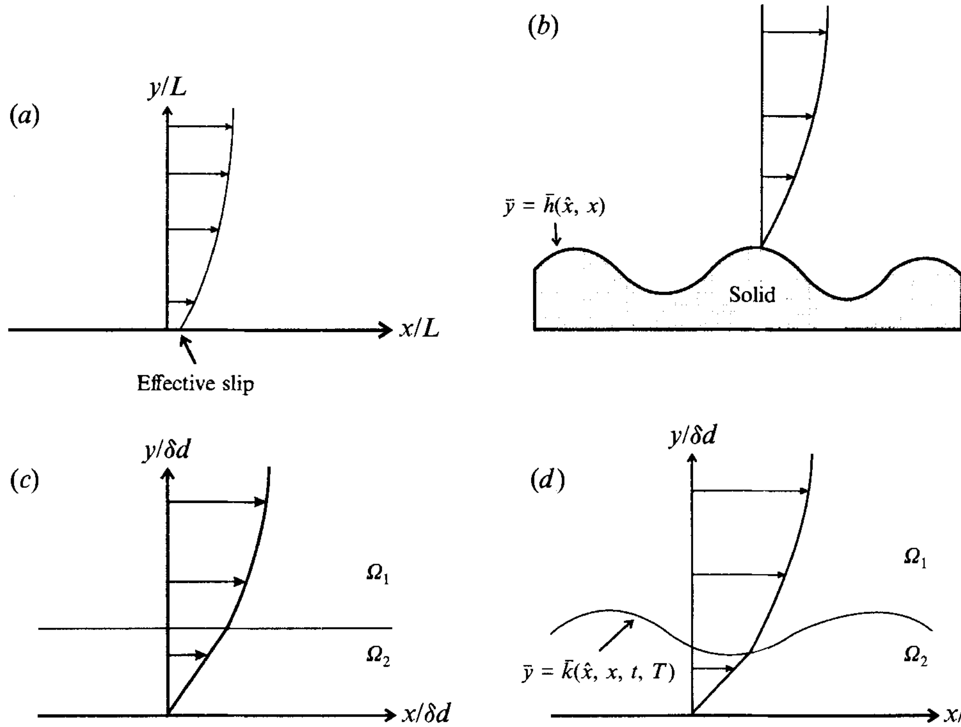

FIGURE 2. Macroscopic model of flow over a coated rough surface as the sum of the flow of a single-phase fluid over a rough surface plus the flow over a horizontal steady fluid/fluid interface plus the flow over a dynamic fluid/fluid interface. (*a*) Slip on the macroscopic scale. The rough surface appears planar. (*b*) Microscopic (inner) single-phase flow over a rough no-slip surface, $y = \epsilon\delta\bar{h}(\hat{x}, x)$. (*c*) Microscopic (inner) flow of two fluids over a smooth surface. The interface between the two fluids is planar. (*d*) Microscopic (inner) flow of two fluids over a smooth surface with a dynamic interface, $y = \epsilon\delta\bar{k}(\hat{x}, x, t, T)$.

condition. From (4.38) we see that the slip coefficient is composed of three terms (see figure 2). The first is due to the surface roughness that arises from single-phase flow over a rough surface (figure 2*b*). The second is proportional to the average thickness of the film and arises from a steady shear flow over a constant-thickness lubricating film lying on a smooth surface (figure 2*c*). The third depends on the dynamics of the interface and this dependence is found to be inversely proportional to the capillary number (figure 2*d*).

By using the method of matched asymptotic expansions and the method of multiple scales we are able to define explicitly the slip coefficient. The effective boundary conditions are derived in the limit of ϵ tending to zero but we needed to assume that the amplitude of the roughness is order $\epsilon\delta$.

How does one solve a problem with this slip coefficient? Although the slip coefficient is given explicitly by (4.39), it appears that in general a sequence of macroscopic (outer) and microscopic (inner) problems must be solved in order to obtain a macroscopic-flow problem that includes the presence of surface roughness and a coating film. One would first solve the Navier–Stokes equations for flow over a no-slip smooth boundary; this identifies the coefficient A. Equation (4.36) then needs to be solved on the microscopic scale for $\bar{k}_0$ with the result substituted into (5.1) and averaged. This determines the slip coefficient and identifies another macroscopic-flow problem over a smooth surface, i.e. the linearized (about the leading-order outer problem) Navier–

Stokes equations for the first correction to the macroscopic-flow field with the effect of the surface roughness and a coating film entering the problem through boundary condition (3.17). Note that although the slip coefficient is used here, the Navier slip condition is not necessary because two macroscopic (outer) problems are solved. So although we have simplified the macroscopic problem, which includes the effects of the roughness and a coating film, we are not permitted to use a Navier slip condition (3.18) which would allow us to solve a single macroscopic problem. The identification of the slip coefficient and the replacement of the no-slip boundary condition by a Navier slip condition (3.18) are useful when the slip coefficient decouples (or couples explicitly) from the leading-order macroscopic (outer) problem. In §5 we found several limits of the parameters where this is possible because the slip coefficient is explicitly known in terms of the initial conditions or else it could easily be determined from them. In particular this is true for single-phase flow over a rough surface where the slip coefficient equals the mean surface amplitude modulation. But when a coating fluid is present, this uncoupling can only be done in certain limiting cases.

This research was supported in part by Department of Energy grants DE-FG02-88ER13927 for M.J.M. and DE-FG02-86ER13641 for S.H.D.

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